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A DECOMPOSITION THEOREM FOR m-CONVEX SETS

BY
MARILYN BREEN

ABSTRACT

Let S be a closed m-convex subset of the plane, $m \geqq 2$, Q the set of points of local nonconvexity of S, with $\text{conv}\, Q \subseteq S$. If there is some point p in $[(\text{bdry}\, S) \cap (\ker S)] \sim Q$, then S is a union of $m - 1$ closed convex sets. The result is best possible for every m.

1. Introduction

Let S be a subset of R^d. The set S is said to be m-convex, $m \geqq 2$, if and only if for every m distinct points in S, at least one of the line segments determined by these points lies in S. A point x in S is said to be a *point of local convexity of* S if and only if there is some neighborhood N of x such that if $y, z \in S \cap N$, then $[y, z] \subseteq S$. If S fails to be locally convex at some point q in S, then q is called a *point of local nonconvexity* (lnc point) *of* S.

Several interesting decomposition theorems have been obtained for closed, planar, 3-convex sets. Valentine [6] has proved that a closed, planar, 3-convex set may be written as a union of three or fewer closed convex sets. Stamey and Marr [4], investigating sets expressible as a union of two convex sets, have obtained the following result: For S closed, bounded, planar, and 3-convex, if S has some point of local convexity in $(\text{bdry}\, S) \cap (\ker S)$, then S is a union of two closed convex sets.

General theorems concerning this type of decomposition for closed m-convex subsets of the plane promise to require a large and fairly unmanageable collection of convex sets, and there have been few results in this area. Guay [1] has proved that if S is a closed, starshaped 4-convex subset of the plane whose kernel is one-dimensional, then S is the union of four convex sets. However, examples by Kay and Guay [2, ex. 4] show that a closed, starshaped m-convex subset of the plane need not be the union of m convex sets for $m > 4$. Thus the

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problem considered in this paper is that of obtaining conditions under which an analogue of the Valentine, Stamey–Marr results might be proved for arbitrary m.

The following familiar terminology will be used: For x, y in S, we say *x sees y via S* if and only if the corresponding segment $[x, y]$ lies in S. Points $x_1, \cdots, x_n$ in S are *visually independent via S* if and only if for $1 \leqq i < j \leqq n$, x_i does not see x_j via S. Throughout the paper, conv S, ker S, and bdry S will be used to denote the convex hull of S, the kernel of S, and the boundary of S, respectively.

2. The decomposition theorem

The following theorem by Lawrence, Hare and Kenelly [3, th. 2] will be useful in obtaining a decomposition for m-convex sets.

LAWRENCE, HARE, KENELLY THEOREM. *Let T be a subset of a linear space such that each finite subset $F \subseteq T$ has a k-partition, $\{F_1, \cdots, F_k\}$, where* $\operatorname{conv} F_i \subseteq T$, *$1 \leqq i \leqq k$. Then T is a union of k convex sets.*

THEOREM 1. *Let S be a closed m-convex subset of the plane, $m \geqq 2$, Q the set of lnc points of S, with* $\operatorname{conv} Q \subseteq S$. *If there is some point p in $[(\text{bdry}\, S) \cap (\ker S)] \sim Q$, then S is expressible as a union of $m - 1$ closed convex sets. The result is best possible for every m.*

PROOF. Assume that S is not convex, for otherwise the result is trivial. Thus $Q \neq \emptyset$ by a theorem of Tietze [5]. We begin with a series of observations which simplify the problem. Since $\operatorname{conv} Q \subseteq S$ and $p \in \ker S$, it follows that $\operatorname{conv}(Q \cup \{p\}) \subseteq S$ and $Q \subseteq \text{bdry conv}\, Q$. Moreover, since $p \in \text{bdry}\, S$, $p \notin \text{int conv}\, Q$, and $Q \cup \{p\} \subseteq \text{bdry conv}\,(Q \cup \{p\})$.

Now since $p \notin Q$, we may select some neighborhood N of p with $N \cap S$ convex. Let H be a line supporting $N \cap S$ at p and let R_1, R_2 be the corresponding closed rays at p with $R_1 \cup R_2 = H$. It is clear that since $p \in \ker S$, S lies in one of the closed halfplanes determined by H. Consider the family $\mathscr{R}$ of rays consisting of R_1, R_2, together with rays of the form $R\ (p, q)$ emanating from p through q for some q in Q. It is not hard to show that for R in $\mathscr{R}$, R contains at most two members of Q. Any two (not necessarily distinct) rays in $\mathscr{R}$ bound a closed subset of S, and we let $\mathscr{W}$ denote the collection of all these closed regions. Moreover, since Q is closed, to every point of S there corresponds a minimal member A of $\mathscr{W}$ which contains x (i.e., if B is in $\mathscr{W}$ and B contains x, then $A \subseteq B$).

Using these remarks, together with the Lawrence, Hare, Kenelly Theorem, it is easy to show that we may assume Q to be finite: For any finite subset $\{x_i : 1 \leqq i \leqq n\}$ of S, to each x_i there corresponds a minimal member A_i of $\mathscr{W}$ which contains x_i. Each lnc point of S in A_i must lie on one of the bounding rays of A_i; hence A_i contains at most four members of Q. The set $S_0 \equiv \mathrm{conv}(Q \cup \{p\}) \cup (\bigcup_{i=1}^{n} A_i)$ is a closed subset of S satisfying the hypothesis of Theorem 1 and having finitely many lnc points. Furthermore, by the Lawrence, Hare, Kenelly Theorem, it is sufficient to show that S_0 is expressible as a union of $m-1$ convex sets. Hence without loss of generality we may assume that Q is finite.

For convenience, order the points of $Q \cup \{p\}$ in a clockwise direction along bdry conv $(Q \cup \{p\})$, letting p be the first point in our ordering. If $Q \cup \{p\} = \{s_1, \cdots, s_n\}$, this induces a natural order among the rays of $\mathscr{R}$, where R_1, R_{n+1} denote the first and last rays, with $R_1 \cup R_{n+1} = H$, and $R_i = R(p, s_i)$ for $2 \leqq i \leqq n$. If B_i denotes the member of $\mathscr{W}$ determined by R_i and R_{i+1}, we let $p_i = s_i$ and $q_i = s_{i+1}$, and refer to p_i, q_i as the members of $Q \cup \{p\}$ corresponding to B_i, $1 \leqq i \leqq n$ (where $s_{n+1} = s_1 = p$).

We examine the B_i sets, B_i determined by consecutive rays R_i, R_{i+1}. For x, y in $B'_i \equiv B_i \sim \{R_i \cup R_{i+1}\}$, $[p, x] \cup [p, y] \subseteq S$, no lnc point of S lies in conv$\{p, x, y\}$, so by a result of Valentine [7, cor. 1], conv$\{p, x, y\} \subseteq S$. Hence $[x, y] \subseteq S$. Then clearly $[x, y] \subseteq B'_i$, B'_i is convex, and cl(B'_i) is convex. We call cl$(B'_i) \equiv V_i$ a *wedge* of S, $1 \leqq i \leqq n$.

Moreover, we may assume that each point of S lies in some wedge: Clearly for x in S not in any wedge, then x must lie on some ray $R(p, q)$, $q \in Q$ (and there are finitely many such rays since Q is finite). Furthermore, for an appropriate q, the open ray $R(q, x) \sim \{q\}$ is disjoint from cl(int S). Let $\mathscr{R}_0$ denote the collection of all such open rays, $\mathscr{L}_0$ the corresponding collection of lines determined by rays in $\mathscr{R}_0$, with card $\mathscr{L}_0 = k$.

We assert that $2 \leqq m - k$ and that the set $S \sim (\cup \mathscr{R}_0)$ is $(m-k)$-convex: Clearly there are k points of S in $\cup \mathscr{L}_0$ which are visually independent via S and which see no point of $S \sim (\cup \mathscr{L}_0) \neq \emptyset$. Hence $S \sim (\cup \mathscr{L}_0)$ contains at most $m-k-1$ points which are visually independent via $S \sim (\cup \mathscr{R}_0)$, and $1 \leqq m-k-1$. Using a standard convergence argument, the set $S \sim (\cup \mathscr{R}_0)$ is $(m-k)$-convex. Also, $S \sim (\cup \mathscr{R}_0)$ satisfies the hypothesis of Theorem 1 and has the property that each of its points lies in a wedge. If we show the set $S \sim (\cup \mathscr{R}_0)$ to be expressible as a union of $m-k-1$ convex sets, these sets, together with the k convex sets of the form $L \cap S$, L in $\mathscr{L}_0$, yield a decomposition of S into $m-1$ convex sets. Thus it is sufficient to prove the theorem for the case in which

S is the union of its wedges. Moreover, this condition implies that each R in $\mathcal{R} \sim \{R_1, R_{n+1}\}$ contains exactly one member of Q, and $R_1 \cap Q = R_{n+1} \cap Q = \emptyset$.

Finally, we are ready for the proof of the theorem. We decompose S into $m-1$ convex sets by defining $\mathcal{U}_1, \cdots, \mathcal{U}_{m-1}$, each an appropriate collection of wedges of S. We assign wedges to the $\mathcal{U}_i$ sets in the following manner:

Using notation discussed above, let $V_1, \cdots, V_n$ denote the wedges of S, where for $i < j$, V_i precedes V_j in our clockwise ordering. Also, assume $2 \leqq m-1 < n$, for otherwise the result is immediate. Clearly $\operatorname{conv}(V_1 \cup V_2) \not\subseteq S$, so let V_1 be in $\mathcal{U}_1$, V_2 be in $\mathcal{U}_2$. If $\operatorname{conv}(V_1 \cup V_3) \subseteq S$, let V_3 be in $\mathcal{U}_1$. Otherwise, let V_3 be in $\mathcal{U}_3$. Inductively, assume that each of the wedges $V_1, \cdots, V_{j-1}$, $j \leqq n$, has been assigned to one of the sets $\mathcal{U}_1, \cdots, \mathcal{U}_k$ and let W_i denote the last wedge assigned to $\mathcal{U}_i$ (i.e., the wedge assigned to $\mathcal{U}_i$ which has largest subscript), $1 \leqq i \leqq k$. If necessary, relabel the W_i and corresponding $\mathcal{U}_i$ sets so that for $1 \leqq i < l \leqq k$, W_i precedes W_l in our ordering. We assign V_j in the following manner: If for some W_i, $1 \leqq i \leqq k$, $\operatorname{conv}(V_j \cup W_i) \subseteq S$, select the W_{i_0} having largest subscript for which $\operatorname{conv}(V_j \cup W_{i_0}) \subseteq S$, and let V_j be in $\mathcal{U}_{i_0}$. Otherwise, let V_j be in $\mathcal{U}_{k+1}$.

We will show that for $k = m-1$, there is a W_i with $\operatorname{conv}(V_j \cup W_i) \subseteq S$. Thus continuing inductively, every wedge may be assigned to some $\mathcal{U}_i$, $1 \leqq i \leqq m-1$. Moreover, it is easy to see that $S_i \equiv \operatorname{conv}(\cup\{V\colon V \text{ a member of } \mathcal{U}_i\})$ is contained in S, and $S = \bigcup_{i=1}^{m-1} S_i$ is the desired decomposition. Note that when S is exactly m-convex, $m-1$ of the $\mathcal{U}$ sets will be required. Certainly the bound of $m-1$ is best possible.

The following result will be useful in completing the proof.

Assertion. *If wedges have been assigned to each of $\mathcal{U}_1, \cdots, \mathcal{U}_k$ in the manner indicated above, then there is a corresponding set of k visually independent points in S.*

Proof of Assertion. Relabel the $\mathcal{U}$ sets so that for $1 \leqq i \leqq k-1$, the first wedge assigned to $\mathcal{U}_i$ precedes the first wedge assigned to $\mathcal{U}_k$ in our clockwise ordering, and let W_k denote the first wedge in $\mathcal{U}_k$. Now examine the wedges preceding W_k. Each of these wedges lies in one of $\mathcal{U}_1, \cdots, \mathcal{U}_{k-1}$, and we relabel these $\mathcal{U}$ sets so that for $2 \leqq i \leqq k-1$, the last member of $\mathcal{U}_i$ (preceding W_k) follows the last member of $\mathcal{U}_1$, and let W_1 denote this last member of $\mathcal{U}_1$. Note that each wedge between W_1 and W_k (i.e., following W_1 and preceding W_k in our clockwise ordering) lies in one of $\mathcal{U}_2, \cdots, \mathcal{U}_{k-1}$.

By the Lawrence, Hare, Kenelly Theorem, we may assume that each wedge has polygonal boundary, and clearly our clockwise ordering imposes a natural order on the boundary of each wedge. Letting p_i, q_i denote the members of

$Q \cup \{p\}$ corresponding to W_i, we will show that there exist points $r_1, x_2, \cdots, x_{k-1}, r_k$ in (bdry S) $\sim Q$ having the following properties: The point $r_1 \neq q_1$ lies on the last segment of (bdry S) $\cap W_1$, $r_k \neq p_k$ on the first segment of (bdry S) $\cap W_k$. For $2 \leqq i \leqq k - 1$, x_i lies either on the line $L \equiv L(q_1, p_k)$ determined by q_1 and p_k or in the open halfspace determined by L which does not contain p (i.e., beyond L from p). Also, x_i lies in a wedge between W_1 and W_k. Moreover, for any y_1 in $[r_1, q_1)$ and any y_k in $(p_k, r_k]$, the points $y_1, x_2, \cdots, x_{k-1}, y_k$ are visually independent via S.

The proof is by induction. For $k = 1$, the result is trivial. For $k = 2$, let S_{12} denote the union of wedges in S between W_1 and W_2, together with W_1 and W_2. Certainly conv $(W_1 \cup W_2) \not\subseteq S$, for otherwise, since W_2 is the first member of $\mathcal{U}_2$ in S_{12} and W_1 is the last member of $\mathcal{U}_1$ preceding W_2, W_2 would have been placed in $\mathcal{U}_1$ in our assignment of wedges. Moreover, it is clear that W_1, W_2 must be consecutive wedges, $q_1 = p_2$, and for any $r_1 \neq q_1$ on the last segment of (bdry S) $\cap W_1$ and any $r_2 \neq p_2$ on the first segment of (bdry S) $\cap W_2$, no point of $[r_1, q_1)$ sees any point of $(p_2, r_2]$ via S.

Let k be greater than 2 and assume that the result is true for all positive integers less than k to prove for k. By the argument above, conv $(W_1 \cup W_k)$ is not contained in S. Hence for some x_1 in W_1 and some x_k in W_k, $[x_1, x_k] \not\subseteq S$. Moreover, we may assume that x_1 is on the last segment of (bdry S) $\cap W_1$, $x_1 \neq q_1$, and that x_k is on the first segment of (bdry S) $\cap W_k$, $x_k \neq p_k$. (Since $k > 2$, $q_1 \neq p_k$.) Further, clearly at least one of x_1, x_k is beyond L (q_1, p_k) from p. Hence there are two cases to consider.

Case 1. If x_1 is beyond L (q_1, p_k) from p, let T_{1k} denote the union of the wedges of S between W_1 and W_k. Then no point of $[x_1, q_1)$ sees any point of T_{1k} beyond L (x_1, q_1) from p, and no point of $[x_1, q_1)$ sees any point of $T_{1k} \sim \{q_1\}$ on or beyond L (q_1, p_k). Also, we may assume that x_k has been selected so that no point on $[x_1, q_1)$ sees any point of $(p_k, x_k]$ via S. Clearly each wedge in T_{1k} belongs to one of $\mathcal{U}_2, \cdots, \mathcal{U}_{k-1}$. Relabel the sets $\mathcal{U}_2, \cdots, \mathcal{U}_{k-1}$ so that the last wedge of $\mathcal{U}_2$ in T_{1k} precedes the last wedge of $\mathcal{U}_i$ in T_{1k}, $3 \leqq i \leqq k - 1$, and let W_2 denote this last wedge of $\mathcal{U}_2$. By our induction hypothesis applied to $T_{1k} \cup \{W_k\}$ and $\mathcal{U}_2, \cdots, \mathcal{U}_k$, we may select r_2, r_k, and (if $3 < k$) $x_3, \cdots, x_{k-1}$ in the manner indicated, with each x_i either on or beyond L (q_2, p_k) and between W_2 and W_k. Letting $x_2 = r_2$, the points $x_2, x_3, \cdots, x_{k-1}$ all lie in T_{1k}, either on L (q_1, p_k) or beyond the line from p, so no point of $[x_1, q_1)$ sees any of these points via S. Also, no point of $[x_1, q_1)$ sees any point of $(p_k, x_k] \cap (p_k, r_k] = (p_k, r'_k]$. Letting $r_1 = x_1$, the points $r_1, r'_k, x_2, \cdots, x_{k-1}$ satisfy the desired requirements, finishing the inductive argument for case 1.

Case 2. In case x_1 is not beyond $L\ (q_1, p_k)$ from p, then x_k must have this property. By an argument similar to that in case 1, no point of $(p_k, x_k]$ sees any point of $T_{1k} \sim \{p_k\}$ on or beyond $L\ (q_1, p_k)$, nor any point on $[x_1, q_1)$. Each wedge between W_1 and W_k lies in one of $\mathcal{U}_2, \cdots, \mathcal{U}_{k-1}$, and we may relabel the $\mathcal{U}$ sets so that the first wedge of $\mathcal{U}_{k-1}$ in T_{1k} follows the first wedge of $\mathcal{U}_i$ in T_{1k}, $2 \leqq i \leqq k-2$. Let W_{k-1} denote this first wedge of $\mathcal{U}_{k-1}$. Applying our inductive hypothesis to $T_{1k} \cup \{W_1\}$ and $\mathcal{U}_1, \cdots, \mathcal{U}_{k-1}$, we may select r_1, r_{k-1}, and (if $3 < k$) $x_2, \cdots, x_{k-2}$ as indicated. Letting $[x_1, q_1) \cap [r_1, q_1) = [r_1', q_1)$, with $x_{k-1} = r_{k-1}$ and $r_k = x_k$, the points $r_1', r_k, x_2, \cdots, x_{k-1}$ satisfy the required conditions. This finishes case 2 and completes the inductive argument. Thus the assertion is proved.

Hence if wedges have been assigned to each of $\mathcal{U}_1, \cdots, \mathcal{U}_k$, there corresponds a set of k visually independent points. Since S is m-convex, no more than $m-1$ of its points are visually independent, and every wedge will be assigned to some $\mathcal{U}_i$, $1 \leqq i \leqq m-1$.

An easy induction may be used to show that $S_i \equiv \text{conv}(\cup\{V\colon V$ a member of $\mathcal{U}_i\}$ is contained in S for $1 \leqq i \leqq m-1$, so $S = \bigcup_{i=1}^{m-1} S_i$ is the desired decomposition. Since S is m-convex, the result is best possible for every m, and the proof of Theorem 1 is complete.

The author wishes to thank the referee for the following observation: Theorem 1 remains valid if instead of the existence of a p in $[(\text{bdry}\, S) \cap (\ker S)] \sim Q$, one assumes that S is (locally) supported at a point p of $(\text{bdry}\, S) \cap (\ker S)$.

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MATHEMATICS DEPARTMENT
UNIVERSITY OF OKLAHOMA
NORMAN, OKLAHOMA 73069, USA